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Midterm HW

Math 565C
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1. Show that τ is an optional time of $(\mathcal{F}_t)$ iff it is a stopping time of $(\mathcal{F}_{t+})$, where

$$\mathcal{F}_{t+} = \bigcap_{u>t} \mathcal{F}_u \tag{1}$$

Solution. ($\implies$) Let τ be an optional time of $\mathcal{F}_t$, then:

$$\{\tau < t\} \in \mathcal{F}_t \tag{2}$$

Let $n \to \infty$

$$\{\tau \le t\} = \bigcap_{n=1}^{\infty} \{\tau < b + \frac{1}{n}\} \in \bigcap_{u>b+\frac{1}{n}} \mathcal{F}_u = \mathcal{F}_{t+} \tag{3}$$

Therefore, τ is a stopping time of $\mathcal{F}_{t+}$.

($\Longleftarrow$) Let τ be a stopping time of the filtration $\mathcal{F}_{t+}$, then:

$$\{\tau \le t\} \in \bigcap_{u<t} \mathcal{F}_u = \mathcal{F}_{t+} \tag{4}$$

Let $n \to \infty$

$$\{\tau < t\} = \bigcup_{n=1}^{\infty} \{\tau < b - \frac{1}{n}\} \in \mathcal{F}_{(b-\frac{1}{n})+} \subset \mathcal{F}_t \tag{5}$$

Therefore, τ is an optional time of $\mathcal{F}_t$.

Therefore, τ is an optional time of $(\mathcal{F}_t)$ iff it is a stopping time of $(\mathcal{F}_{t+})$.

2. Hitting times
a.) Let ξ_t, $t \ge 0$ with values in $\mathbb{R}^d$ (or in any metric space) whose paths are continuous. Let Γ be a closed set. Prove that the random variable $\tau_\Gamma = \min\{t : \xi_t \in \Gamma\}$ (by definition, $\tau_\Gamma = \infty$ if $\xi_t \notin \Gamma$ for any t) is a stopping time wrt the filtration $\mathcal{F}_t$ associated with the process $\mathcal{F}_t = \sigma(\xi_s : s \le t)$.

Proof. Consider

$$\{\tau_\Gamma \le t\} = \bigcup_{u' \le t} \{\xi_s \in \Gamma \ \forall s \in [u', t]\} \ u' \in \mathbb{Q} \tag{6}$$

Equality is not immediate. Let $\omega \in \Omega$. $\tau(\omega) \le t$. Choose u' st $u \le t \in \mathbb{Q}$, so $\tau_\Gamma(\omega) \le u' \le t$ and note that $\xi_s(\omega) \in \Gamma \ \forall u' \le s \le t$. But, since s can assume any value, the above event $\{\tau_\Gamma \leq t\}$ cannot be represented by a countable union yet. However, the following representation can be made

$$\{\xi_s \in \Gamma \ \forall s \in [u', t]\} = \bigcap_{u' \leq s \leq t} \{\xi_s \in \Gamma\} \ s \in \mathbb{Q} \tag{7}$$

which is closed. So now we have:

$$\bigcup_{u' \leq t} \bigcap_{u' \leq s \leq t} \{\xi_s \in \Gamma\} \ u', s \in \mathbb{Q} \tag{8}$$

which is a countable union of a countable intersection. This implies $\{\tau_\Gamma(\omega) < t\} \in \mathcal{F}_t$ since the filtration is made up of past events and the above event is just such an event. $\square$

b.) Show an example where the analogous statement is no longer true when Γ is an open set.

Solution.

c.) Prove that if Γ is open, the hitting time τ_Γ is a stopping time wrt the filtration $(\mathcal{F}_{t+})$.

Proof. Consider the event $\{\tau_\Gamma > t\} = \{\tau_\Gamma \leq t\}^c$. Fix some $u > t$. Need to show that $\{\tau_\Gamma > t\} \in \mathcal{F}_u$. Consider the relation:

$$\{\tau_\Gamma > t\} = \bigcup_{u \geq u' > t} \{\xi_s \in \Gamma^c \ \forall s \leq u'\} \ u' \in \mathbb{Q} \tag{9}$$

Equality is not immediate. Let $\omega \in \Omega$. $\tau(\omega) > t$. Choose u' st $u > u' \in \mathbb{Q}$, so $\tau_\Gamma(\omega) > u' > t$ and note that $\xi_s(\omega) \in \Gamma^c \ \forall s \leq u'$. But, since s can assume any value, the above event $\{\tau_\Gamma > t\}$ cannot be represented by a countable union yet. However, the event on the RHS of the above expression can be represented by:

$$\{\xi_s \in \Gamma^c \ \forall s \leq u'\} = \bigcap_{s \leq u'} \{\xi_s \in \Gamma^c\} \ s \in \mathbb{Q} \tag{10}$$

which is a closed set. So now we have:

$$\bigcup_{u \geq u' > t} \bigcap_{s \leq u'} \{\xi_s \in \Gamma^c\} \ u', s \in \mathbb{Q} \tag{11}$$

which is a countable union of a countable intersection. This implies $\{\tau_\Gamma(\omega) > t\} \in \bigcap_{u>t} \mathcal{F}_u$. Since $\bigcap_{u>t} \mathcal{F}_u$ is a filtration, the complement $\{\tau_\Gamma(\omega) \leq t\} \in \bigcap_{u>t} \mathcal{F}_u$. Therefore, τ_Γ is a stopping time of the filtration $\mathcal{F}_{t+}$. $\square$

3. Background on stopping and optional times
   a.) Prove that if σ and τ are stopping times of the filtration $(\mathcal{F}_t)$. then

Proof.

$$\{\sigma \leq t\} \cup \{\tau \leq t\} \in \mathcal{F}_t \implies \{\sigma \vee \tau \leq t\} \in \mathcal{F}_t \tag{12}$$
$$\{\sigma \leq t\} \cap \{\tau \leq t\} \in \mathcal{F}_t \implies \{\sigma \wedge \tau \leq t\} \in \mathcal{F}_t \tag{13}$$

For the sum $\sigma + \tau$, look at the complement of the desired event for an arbitrary $\omega \in \Omega$:

$$\{\sigma(\omega) + \tau(\omega) > t\} \in \mathcal{F}_t \tag{14}$$

Define the following event

$$E = (\bigcup_{r \leq 0, r \in \mathbb{Q}} \{\sigma(\omega) > \tau(\omega) - r\} \cap \{\tau(\omega) > r\}) \cup (\{\sigma(\omega) > t\} \cup \{\tau(\omega) > t\}) \tag{15}$$

Although not immediately clear, $E \subset \{\sigma(\omega) + \tau(\omega) > t\}$. From measure theory, we know that if two events, a, b are measurable wrt some σ-algebra $\mathcal{F}_t$, then $a + b$ is measurable wrt $\mathcal{F}_t$ iff $\exists$ some $r \in \mathbb{Q}$ st $a > t - r$ $b > r$. So now, consider $\forall \delta > 0$

$$\begin{aligned} \tau(\omega) &= \delta & (16) \\ \sigma(\omega) &= t - \delta & (17) \end{aligned}$$

Now, $\forall \delta$ $\exists r \in \mathbb{Q}$ st $\tau(\omega) < r$ and $\sigma(\omega) > t - r$. This implies, $\{\sigma(\omega) + \tau(\omega) > t\}$ is measurable wrt to the filtration $\mathcal{F}_t$. Therefore, the complement of the event, $\{\sigma(\omega) + \tau(\omega) \leq t\} \in \mathcal{F}_t$. □

b.) Let τ be a stopping time of $\mathcal{F}_t$. We define an event A to be measurable at time τ if for every $t \geq 0$, $A \cup \{\tau \leq t\} \in \mathcal{F}_t$. Prove that the collection $\mathcal{F}_\tau$ of all such events is a σ-algebra and that the random variable τ is measurable wrt this σ-variable. Show that when $\tau \equiv t$ is constant, then $\mathcal{F}_\tau = \mathcal{F}_t$.

Proof. To show that the collection of all events A is a σ-algebra, need to show that the union of all such events $\in \mathcal{F}_t$ and that A^c, the complement of A, is $\in \mathcal{F}_t$ as well. The union is easy, since:

$$(\bigcup_{i=1}^{n} A_i) \cap \{\tau \leq t\} = \bigcup_{i=1}^{n} (A_i \cap \{\tau \leq t\}) \in \mathcal{F}_t \tag{18}$$

The intersection requires the definition of τ as a stopping time. Since τ is a stopping time, $\{\tau \leq t\} \in \mathcal{F}_t$. Since $A \cap \{\tau \leq t\} \in \mathcal{F}_t$:

$$(A \cap \{\tau \leq t\}) \cup \{\tau > t\} \in \mathcal{F}_t \tag{19}$$
$$\tag{20}$$

Take the complement

$$\begin{aligned} [(A \cap \{\tau \leq t\} \cup \{\tau > t\}]^c &= (A \cap \{\tau \leq t\})^c \cap \{\tau \leq t\} & (21) \\ &= (A^c \cup \{\tau > t\}) \cap \{\tau \leq t\} \in \mathcal{F}_t & (22) \end{aligned}$$

But, the event $\{\tau > t\}$ is not necessarily in the filtration $\mathcal{F}_t$, therefore $A^c \in \mathcal{F}_t$ necessarily. So, the collection of events A is a σ-algebra. □

c.) Let τ be an optional time of $\mathcal{F}_t$. McKean defines the σ-algebra $\mathcal{F}_{t+}$ as the collection of all events A such that $A \cap \{\tau < t\} \in \mathcal{F}_t$ (note the strict inequality). Prove that this is a σ-algebra and that τ is measurable wrt it. As a consequence, show that if $\mathcal{F}_t = \mathcal{B}_t$ is the Brownian filtration and τ is as above, then $b(\tau)$ is measurable wrt $\mathcal{B}_{t+}$. Since τ may be equal to ∞, this requires defining $b(\infty)$ as any constant or, as McKean does, as ∞.

Proof. ☐

4. Strong Markov property of Brownian Motion

a.) Argue that, including in addition independence of $(\mathcal{B}_{t+})$, the developed argument is equivalent to the statement that McKean actually proves.

Solution.

b.) With this setup and explanations, you should now be able to complete the proof. The main technical idea is to approximate τ by stopping times with dyadic rational values and use the assumption on B to factor the expectation in the last line on page 10.

Proof. ☐

c.) Use Dynkin-Hunt's result to prove Blumenthal's 01 law: $\Pr(B) = 0$ or 1 if $B \in \mathcal{B}_{0+}$.

Proof. Dynkin-Hunt's statement is as follows:

If τ is a stopping time conditional on $\tau < \infty$, $b^+(t) \equiv b(t + \tau) - b(\tau) : \ t \geq 0$ is a Brownian motion, independent of $b(t) : \ t \leq \tau+$, i.e. independent of $\mathcal{B}_{t+}$.

Having established this, we can prove Blumenthal's 01 law.

Let $\tau = 0$. $b^+(t) = b(t)$. So, by Dynkin-Hunt's statement, B is independent of itself and measurable over both b^+ and $\mathcal{B}_{0+}$. Therefore:

$$Pr(B) = Pr(B \cap B) = Pr(B)Pr(B) = Pr(B)^2 \tag{23}$$

Therefore, $Pr(B) = 0$ or 1 if $B \in \mathcal{B}_{0+}$. ☐

d.) As an application of Blumenthal's law, show that with probability one the Brownian path returns to 0 infinitely many times on any time interval $[0, \epsilon]$.

Solution. (I could not figure out how to do this using Blumenthal's law, however the following argument should work)
First, need to show that $\forall \epsilon > 0$ there is at least one 0 in the interval. This is easy, since the Brownian motion b_t assumes both positive and negative values on any interval $(0, \epsilon)$ with probability 1. Since b_t is continuous, apply the Intermediate Value Theorem,

which states that $b_t = 0$ for some $t \in (0, \epsilon)$. Now need to show that there are infinitely many. This follows immediately from the above conclusion. For example, let $t_1 \in (0, \epsilon)$ and $b_{t_1} = 0$. Now, pick any $k \in \mathbb{N}$ such that $t_1 > \frac{\epsilon}{k}$. From the above argument, $\exists t_2 \in (0, \frac{\epsilon}{k})$ such that $b_{t_2} = 0$. This process can be repeated infinitely many times. Therefore, b_t has infinitely many zeros in any interval $(0, \epsilon)$.

5. The zero set of Brownian motion
Define the zero set of the Brownian path by:

$$Z = \{t : b_t = 0\} \tag{24}$$

Part d) of the last problem proves that with probability one Z is infinite. Here we study Z in more detail.

a.) Prove that with probability one the Lebesgue measure of Z equals 0. Hint: use Fubini theorem.

Proof. Need to look at the expected value of the measure of Z, $(|Z|)$. Let Ω represent the probability space and $1_{t \in Z}$ denote the desired indicator function. For any $t \neq 0$, b_t has normal distribution with mean 0 and variance t; the following holds

$$\begin{aligned} P_x(t \in Z) = P_x(b_t = 0) &= 0 & (25) \\ \implies \int_\Omega 1_{t \in Z} dP_x &= 0 & (26) \end{aligned}$$

where x denotes position of the 1-d Brownian motion. Using Fubini theorem yields:

$$E(|Z|) = \int_\Omega \int_0^\infty 1_{t \in Z} dt dP_x = \int_0^\infty \int_\Omega 1_{t \in Z} dP_x dt = 0 \tag{27}$$

which implies that the measure of Z is 0 almost surely, i.e. with probability 1. □

b.) Use strong Markov property to show that with probability one Z does not contain an open interval.

Solution. Since the Brownian motion repeats itself past every stopping time (strong Markov), we have non-differentiability of every $b^+(t)$ defined as in the previous sense. Therefore, open intervals can not be contained within Z since that would violate non-differentiability.

c.) Prove that with probability one Z has no isolated points. It is thus a closed set, dense in itself. Such sets are called perfect. The Cantor set is a well-known example of a perfect set.

Proof. Let $R_t = \inf\{u > t : b_u = 0\}$, $T_0 = \inf\{u > 0 : b_u\}$. Take the fact that the Brownian motion b_t returns to 0 regularly, this implies $P_x(R_t < \infty) = 1$ for t finite.

Using the strong Markov property of the Brownian motion, the Brownian motion can be shifted and restarted at R_t by θ_{R_t}, take expectation:

$$P_x(T_0 \circ \theta_{R_t} > 0|\mathcal{F}_{R_t}) = P_0(T_0 > 0) = 0 \tag{28}$$

Take expectation again and:

$$P_x(T_0 \circ \theta_{R_t} > 0 \text{ for some } t \in \mathbb{Q}) = 0 \tag{29}$$

So if a point $u = R_t$ for some $t \in \mathbb{Q}$ is isolated from the left, the expectation shows that u is an accumulated point from the right. Therefore, Z has no isolated points. ☐

d.) Prove that with probability one $Z \cap [0,1]$ is homeomorphic to the Cantor set.

Proof. This follows from Brouwer's theorem, which states that any perfect set with compact support is homeomorphic to the Cantor set. I cannot immediately think of a particular homeomorphism that works though. ☐

6. Some stochastic differential equations and Itô calculus.
a.) Show that the process $X_t = \frac{b_t}{1+t}$ solves the stochastic differential equation

$$dX_t = -\frac{1}{1+t}X_t dt + \frac{1}{1+t}db_t \tag{30}$$

Solution. First, define the continuous function $u(t, b_t) = \frac{b_t}{1+t}$ and then find partial derivatives:

$$u_0 = \frac{\partial u}{\partial t} = -\frac{b_t}{(1+t)^2} = -\frac{1}{1+t}u(t, b_t) \tag{31}$$

$$u_1 = \frac{\partial u}{\partial b_t} = \frac{1}{1+t} \tag{32}$$

Use Itô's lemma and replace u with X_t:

$$dX_t = u_0 dt + u_1 db_t \tag{33}$$

$$= -\frac{1}{1+t}X_t + \frac{1}{1+t}db_t \tag{34}$$

Therefore, the given X_t solves the given SDE.

b.) "Brownian motion on an ellipse" : let $X_1(t) = a\cos b_t$ and $X_2(t) = b\sin b_t$, with $a, b > 0$. Show that $X_1(t)$ and $X_2(t)$ satisfy the system of SDEs.

$$dX_1(t) = -\frac{1}{2}X_1(t)dt - \frac{a}{b}X_2(t)db_t \tag{35}$$

$$dX_2(t) = -\frac{1}{2}X_2(t)dt + \frac{b}{a}X_1(t)db_t \tag{36}$$

Solution. Define $u^1(t, b_t) = a\cos b_t,\ u^2(t, b_t) = b\sin b_t$ and then find partials:

$$\begin{align}
u_0^1 &= \frac{\partial u_1}{\partial t} = 0 \tag{37}\\
u_1^1 &= \frac{\partial u_1}{\partial b_t} = -a\sin b_t = -\frac{a}{b}X_2(t) \tag{38}\\
u_{11}^1 &= \frac{\partial^2 u_1}{\partial b_t^2} = -a\cos b_t = -X_1(t) \tag{39}\\
u_0^2 &= \frac{\partial u_1}{\partial t} = 0 \tag{40}\\
u_1^2 &= \frac{\partial u_1}{\partial b_t} = b\cos b_t = \frac{b}{a}X_1(t) \tag{41}\\
u_{11}^2 &= \frac{\partial^2 u_1}{\partial b_t^2} = -b\sin b_t = -X_2(t) \tag{42}\\
& \tag{43}
\end{align}$$

Use Itô's lemma and replace u^1, u^2 with $X_1(t), X_2(t)$:

$$\begin{align}
dX_1(t) &= u_0^1 dt + u_1^1 db_t + \frac{1}{2}u_{11}^1 dt \tag{44}\\
&= -\frac{1}{2}X_1(t) - \frac{a}{b}X_2(t)db_t \tag{45}\\
dX_2(t) &= u_0^2 dt + u_1^2 db_t + \frac{1}{2}u_{11}^2 dt \tag{46}\\
&= -\frac{1}{2}X_2(t) + \frac{b}{a}X_1(t)db_t \tag{47}\\
& \tag{48}
\end{align}$$

Therefore, $X_1(t), X_2(t)$ satisfy the given set of SDEs.

c.) Solve the equations:

$$dX_t = X_t dt + db_t \tag{49}$$

and

$$dX_t = -X_t dt + e^{-t} db_t \tag{50}$$

Solution. $(dX_t = X_t dt + db_t)$
Begin by looking at SDE and extracting partial derivatives of a continuous function $u(t, b_t)$ to be used in Itô's lemma:

$$\begin{align}
u_0 &= u(t, b_t) \tag{51}\\
u_1 &= 1 \tag{52}
\end{align}$$

Elementary ODE theory can be used to find that u_0 leads to a term like e^t, while u_1 leads to a b_t term in the solution. Combine these and add initial condition $X_0 = c$:

$$X_t = ce^t + b_t \tag{53}$$

To check, apply steps taken in parts a.) and b.) using Itô's lemma.

$$\begin{aligned} dX_t &= u_0 dt + u_1 db_t \qquad (54) \\ &= X_t + db_t \quad \clubsuit \qquad (55) \end{aligned}$$

Which matches initial SDE.

$(dX_t = -X_t dt + e^{-t} db_t)$
Begin same as in last equation:

$$\begin{aligned} u_0 &= -u(t, b_t) \qquad (56) \\ u_1 &= e^{-t} \qquad (57) \end{aligned}$$

These two partials lead to 1.) an e^-t and a b_t term multiplied. The only thing that remains is to apply the initial condition. The solution can't have the form cb_te^{-t} since then the term in front of db_t would be ce^-t, not the desired e^-t. Since the Brownian motion at time 0, b_0, is 0, the solution can be formed as:

$$X_t = c + b_t e^{-t} \tag{58}$$

To check, apply Itôs lemma

$$\begin{aligned} dX_t &= u_0 dt + u_1 db_t \qquad (59) \\ &= -X_t dt + e^{-t} db_t \quad \clubsuit \qquad (60) \end{aligned}$$

which matches initial SDE.

d.) Let $a, b \in \mathbb{R}$. Consider the SDE:

$$dY_t = \frac{b - Y_t}{1 - t} dt + db_t; \quad 0 \le t < 1;\ Y_0 = a. \tag{61}$$

Verify that the process

$$Y_t = a(1-t) + bt + (1-t)\int_0^t \frac{db_s}{1-s}; \quad 0 \le t < 1 \tag{62}$$

is a solution of the above equation, satisfying $\lim_{t \to 1} Y_t = b$ with probability one. This is one of the ways of defining an important Gaussian process- the Brownian bridge connecting a to b. Find the mean and the covariance of Y_t for $a = b = 0$ and verify that whenever B_t is a Brownian motion, the process $X_t = B_t - tB_1$, $0 \le t < 1$ has the same distribution as Y_t. The latter is a more standard way of introducing the Brownian bridge.

Solution. Differentiate the given formula for Y_t:

$$\begin{aligned}
dY_t &= -adt + bdt + (1-t)d[\int_0^t \frac{db_s}{1-s}] + \int_0^t \frac{db_s}{1-s} d(1-t) && (63)\\
&= (b-a)dt + (1-t)\frac{db_t}{1-t} + \int_0^t \frac{db_s}{1-s}(-dt) && (64)\\
&= \frac{1-t}{1-t}[(b-a) - \int_0^t \frac{db_s}{1-s}]dt + db_t && (65)\\
&= [\frac{(b-a) - (b-a)t - (1-t)\int_0^t \frac{db_s}{1-s}}{1-t}] + db_t && (66)\\
&= \frac{b - [a(1-t) + bt + (1-t)\int_0^t \frac{db_s}{1-s}]}{1-t} + db_t && (67)\\
&= \frac{b-Y_t}{1-t} + db_t \quad \clubsuit && (68)
\end{aligned}$$

which shows that the given equation is a solution of the SDE.

Let $a = b = 0$ and find the covariance and mean of Y_t. Integrating $Y_t = (1-t)\int_0^t \frac{db_s}{1-s}$ leads to $Y_t = b_t$; where the $\frac{1}{1-s}$ is a removable singularity, due to the $(1-t)$ term. Therefore, the mean and covariance of the distribution for Y_t are simply:

$$\begin{aligned}
E[Y_t] &= E[b_t] = 0 && (69)\\
E[Y_sY_t] &= E[b_sb_t] = s \wedge t && (70)
\end{aligned}$$

Now, need to show that the process $X_t = b_t - tb_1$ is a Brownian motion identical in distribution to Y_t. Need to show that the mean and covariance of the process X_t is the same as those of Y_t. The mean and covariance of X_t are:

$$\begin{aligned}
E[X_t] &= E[b_t - tb_1] = E[b_t] - E[tb_1] = 0 - 0 = 0 && (71)\\
E[X_sX_t] &= E[(b_s - sb_1)(b_t - tb_1)] = E[b_sb_t] = s \wedge t && (72)
\end{aligned}$$

So, X_t and Y_t are identical in distribution.

e.) Let $(b_1(t), b_2(t))$ be a two-dimensional Brownian motion. The complex-valued process $b_t = b_1(t) + ib_2(t)$ is called a complex Brownian motion. Let F be an entire analytic function. For $Z_t = F(b_t)$ prove that $dZ_t = F'(b_t)db_t$. It follows that Z_t is a (complex-valued) martingale.

Proof. Let F be entire. Therefore $\frac{dF}{db_t} = F'$ and all subsequent derivatives are well-defined (and exist) for all arguments b_t. Now, use Itô's lemma with $u(b_1(t), b_2(t)) =$

$F(b_t)$. Find the partial derivatives of u:

$$\begin{aligned} u_0 &= \frac{\partial u}{\partial t} = 0 && (73)\\ u_1 &= \frac{\partial u}{\partial b_1(t)} = F'(b_1(t) + ib_2(t)) = F'(b_t) && (74)\\ u_2 &= \frac{\partial u}{\partial b_2(t)} = iF'(b_1(t) + ib_2(t)) = iF'(b_t) && (75)\\ u_{11} &= \frac{\partial u^2}{\partial b_1^2(t)} = F''(b_1(t) + ib_2(t)) = F''(b_t) && (76)\\ u_{22} &= \frac{\partial u^2}{\partial b_2^2(t)} = i^2F''(b_1(t) + ib_2(t)) = -F''(b_t) && (77) \end{aligned}$$

The mixed partials u_{12} and u_{21} are irrelevant, since $db_i(t)db_j(t) = \delta_{ij}dt$. Now, Itô's lemma yields:

$$\begin{aligned} dZ_t &= u_0dt + \sum_{i=1}^{2} u_i db_i(t) + \frac{1}{2}\sum_{i,j\leq 2} u_{ij}db_i(t)db_j(t) && (78)\\ &= 0dt + F'(b_t)db_1(t) + iF'(b_t)db_2(t) + \frac{1}{2}(F''(b_t)dt - F''(b_t)dt) && (79)\\ &= F'(b_t)(db_1(t) + idb_2(t)) && (80)\\ &= F'(b_t)db_t && (81) \end{aligned}$$

where the substitution $b_t \rightarrow b_1(t) + ib_2(t)$ leads to the relation $db_t = db_1(t) + idb_2(t)$. Done. □

f.) The following is an important Itô formula for an iterated stochastic integral:

$$n!\int_0^t db_{u_n}(\int_0^{u_n} db_{u_{n-1}}(\ldots\int_0^{u_2} db_{u_1})\ldots) = t^{\frac{n}{2}}h_n(\frac{b_t}{\sqrt{t}}) \tag{82}$$

where

$$h_n(x) = (-1)^n e^{\frac{x^2}{2}}\frac{d^n}{dx^n}(e^{\frac{-x^2}{2}}) \tag{83}$$

is the Hermite polynomial of degree n.
Verify the formula for $n = 0, 1, 2, 3$ and then prove it for all n.

Solution. n=0:

$$1 = 0! = t^{\frac{0}{2}}h_0(\frac{b_t}{\sqrt{t}}) = 1 * 1 = 1 \ \clubsuit \tag{84}$$

n=1:

$$b_t = 1!\int_0^t db_{u_1} = t^{\frac{1}{2}}h_1(\frac{b_t}{\sqrt{t}}) = t^{\frac{1}{2}}\frac{b_t}{\sqrt{t}} = b_t \ \clubsuit \tag{85}$$

n=2:

$$2!\int_0^t db_{u_2}\int_0^{u_2} db_{u_1} = 2!\int_0^t b_{u_2}db_{u_2} \tag{86}$$

$$= 2!(\frac{1}{2}(b_t^2-t) = t((\frac{b_t}{\sqrt{t}})^2-1) = t^{\frac{2}{2}}h_2(\frac{b_t}{\sqrt{t}}) \quad \clubsuit \tag{87}$$

n=3:

$$3!\int_0^t db_{u_3}\int_0^{u_3} db_{u_2}\int_0^{u_1} db_{u_1} = 3!\int_0^t db_{u_3}\int_0^{u_3} b_{u_2}db_{u_2} \tag{88}$$

$$= 3\int_0^t (b_{u_3}^2-u_3)db_{u_3} = 3\int_0^t b_{u_3}^2 db_{u_3} - 3\int_0^t u_3 db_{u_3} \tag{89}$$

$$\equiv I \tag{90}$$

Now, perform integration by parts on the first integral, with $u = b_{u_3}$ and $v = b_{u_3}db_{u_3}$ this yields:

$$I = b_t^3 - tb_t - 2\int_0^t u_3 db_{u_3} = b_t^3 - 3tb_t \tag{91}$$

$$= t^{\frac{3}{2}}((\frac{b_t}{\sqrt{t}})^3 - 3\frac{b_t}{\sqrt{t}}) = t^{\frac{3}{2}}h_3(\frac{b_t}{\sqrt{t}}) \quad \clubsuit \tag{92}$$

Now, prove for all n:

Proof. (Induction, obviously) We have already shown that the formula works for $n = 0, 1, 2$ and 3, now assume it works for the k^{th} step (and shorthand the integral as $\int ...$);

$$k!\int ... = t^{\frac{k}{2}}h_k(\frac{b_t}{\sqrt{t}}) \tag{93}$$

Look at the $(k+1)^{th}$ step:

$$(k+1)!\int_0^t db_{u_{k+1}}(\int ...) = (k+1)\int_0^t db_{u_{k+1}}(k!\int ...) \tag{94}$$

$$= (k+1)\int_0^t u_{k+1}^{\frac{k}{2}}h_k(\frac{b_{u_{k+1}}}{\sqrt{u_{k+1}}})db_{u_{k+1}} \equiv II \tag{95}$$

Now, make substitution $b'_{u_{k+1}} = \frac{b_{u_{k+1}}}{\sqrt{u_{k+1}}}$ and use (i) the relationship that $h'_n = nh_{n-1}$ and (ii) the Fundamental Theorem of Calculus; the integral becomes:

$$II = t^{\frac{k+1}{2}}h_{k+1}(b'_t) = t^{\frac{k+1}{2}}h_{k+1}(\frac{b_t}{\sqrt{t}}) \tag{96}$$

This proves the formula $\forall n$. □